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Solving Some Type of Improper Fractional Integral Using Differentiation under Fractional Integral Sign and Integration by Parts for Fractional Calculus

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Abstract: **In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we study some type of improper fractional integral. We can obtain the exact solution of this improper fractional integral by using differentiation under fractional integral sign and integration by parts for fractional calculus. In fact, our result is a generalization of classical calculus result.**

Keywords: **Jumarie's modified R-L fractional calculus, new multiplication, fractional analytic functions, improper fractional integral, differentiation under fractional integral sign, integration by parts for fractional calculus.**

I. INTRODUCTION

In 1695, the concept of fractional derivative first appeared in a famous letter between L'Hospital and Leibniz. Many great mathematicians have further developed this field. We can mention Euler, Lagrange, Laplace, Fourier, Abel, Liouville, Riemann, Hardy, Littlewood, and Weyl. Fractional calculus has important applications in various fields such as physics, mechanics, electrical engineering, biology, economics, viscoelasticity, control theory, and so on [1-11]. However, fractional calculus is different from ordinary calculus. The definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [12-15]. Since Jumarie's modified R-L fractional derivative helps avoid non-zero fractional derivative of constant functions, it is easier to use this definition to associate fractional calculus with classical calculus.

In this paper, based on Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions we study the following improper α-fractional integral:

$$\left({}_{0}I^{\alpha}_{+\infty}\right)\left[E_{\alpha}(-tx^{\alpha})\otimes_{\alpha} sin_{\alpha}(x^{\alpha})\otimes_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(-1)}\right],$$

where $0<\alpha\leq 1$ and $t>0$. The exact solution of this improper α-fractional integral can be obtained by using differentiation under fractional integral sign and integration by parts for fractional calculus. In fact, our result is a generalization of traditional calculus result.

II. PRELIMINARIES

At first, we introduce the fractional calculus used in this paper and its properties.

Definition 2.1 ([16])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\,. \tag{1}$$

where $\Gamma(\ \ )$ is the gamma function. And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\,, \tag{2}$$

Proposition 2.2 ([17])**:** *If α,β,x_0,C are real numbers and $\beta \geq \alpha > 0$, then*

$$\left({}_{0}D_x^{\alpha}\right)\left[x^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{0}D_x^{\alpha}\right)[C] = 0. \tag{4}$$

In the following, we introduce the definition of fractional analytic function.

Definition 2.3 ([18])**:** Let x, x_0, and a_k be real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([19])**:** If $0 < \alpha \leq 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{6}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}\otimes_\alpha \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k}\otimes_\alpha \sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k} \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k}. \end{aligned} \tag{8}$$

Definition 2.5 ([20])**:** Suppose that $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions. Then $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha k} = f_\alpha(x^\alpha)\otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the k-th power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha(-1)}$.

Definition 2.6 ([21])**:** Assume that $0 < \alpha \leq 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes k}. \tag{9}$$

In addition, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes 2k}, \tag{10}$$

and

$$sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes (2k+1)}. \tag{11}$$

Theorem 2.7 ([22])(differentiation under fractional integral sign)**:** *If* $0 < \alpha \le 1$, *t is a nonzero real variable, and* $f_\alpha(x^\alpha)$ *is a* α*-fractional analytic function at* $x = 0$, *then*

$$\frac{d}{dt}\left({}_0I_x^\alpha\right)[f_\alpha(tx^\alpha)] = \left({}_0I_x^\alpha\right)\left[\frac{d}{dt} f_\alpha(tx^\alpha)\right]. \tag{12}$$

Theorem 2.8 (integration by parts for fractional calculus) ([23])**:** *Suppose that* $0 < \alpha \le 1$, a, b *are real numbers, and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ *are* α*-fractional analytic functions, then*

$$\left({}_aI_b^\alpha\right)\left[f_\alpha(x^\alpha) \otimes \left({}_aD_x^\alpha\right)[g_\alpha(x^\alpha)]\right] = [f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha)]_{x=a}^{x=b} - \left({}_aI_b^\alpha\right)\left[g_\alpha(x^\alpha) \otimes \left({}_aD_x^\alpha\right)[f_\alpha(x^\alpha)]\right]. \tag{13}$$

III. MAIN RESULTS

In this section, we use differentiation under fractional integral sign and integration by parts for fractional calculus to solve an improper fractional integral. At first, we need a lemma.

Lemma 3.1: *If* $0 < \alpha \le 1$ b, t *are real numbers. Then*

$$\left({}_0I_b^\alpha\right)[E_\alpha(-tx^\alpha) \otimes_\alpha sin_\alpha(x^\alpha)] = \frac{E_\alpha(-tb^\alpha) \otimes_\alpha (-t sin_\alpha(b^\alpha) - cos_\alpha(b^\alpha))}{1+t^2} + \frac{1}{1+t^2}. \tag{14}$$

Proof By integration by parts for fractional calculus,

$$\left({}_0I_b^\alpha\right)[E_\alpha(-tx^\alpha) \otimes_\alpha sin_\alpha(x^\alpha)]$$

$$= -\left({}_0I_b^\alpha\right)\left[E_\alpha(-tx^\alpha) \otimes_\alpha \left({}_0D_x^\alpha\right)[cos_\alpha(x^\alpha)]\right]$$

$$= -\left({}_0I_b^\alpha\right)\left[E_\alpha(-tx^\alpha) \otimes_\alpha \left({}_0D_x^\alpha\right)[cos_\alpha(x^\alpha)]\right]$$

$$= -[E_\alpha(-tx^\alpha) \otimes cos_\alpha(x^\alpha)]_{x=0}^{x=b} + \left({}_0I_b^\alpha\right)\left[cos_\alpha(x^\alpha) \otimes_\alpha \left({}_0D_x^\alpha\right)[E_\alpha(-tx^\alpha)]\right]$$

$$= -[E_\alpha(-tb^\alpha) \otimes cos_\alpha(b^\alpha)] + 1 + \left({}_0I_b^\alpha\right)\left[cos_\alpha(x^\alpha) \otimes_\alpha [-tE_\alpha(-tx^\alpha)]\right]$$

$$= -[E_\alpha(-tb^\alpha) \otimes cos_\alpha(b^\alpha)] + 1 - t\left({}_0I_b^\alpha\right)\left[E_\alpha(-tx^\alpha) \otimes_\alpha \left({}_0D_x^\alpha\right)[sin_\alpha(x^\alpha)]\right]$$

$$= -[E_\alpha(-tb^\alpha) \otimes cos_\alpha(b^\alpha)] + 1 - t\left[[E_\alpha(-tx^\alpha) \otimes sin_\alpha(x^\alpha)]_{x=0}^{x=b} - \left({}_0I_b^\alpha\right)\left[sin_\alpha(x^\alpha) \otimes_\alpha \left({}_0D_x^\alpha\right)[E_\alpha(-tx^\alpha)]\right]\right]$$

$$= -[E_\alpha(-tb^\alpha) \otimes cos_\alpha(b^\alpha)] + 1 - t\left[E_\alpha(-tb^\alpha) \otimes sin_\alpha(b^\alpha) + t\left({}_0I_b^\alpha\right)[sin_\alpha(x^\alpha) \otimes_\alpha E_\alpha(-tx^\alpha)]\right]$$

$$= \left[E_\alpha(-tb^\alpha) \otimes [-t sin_\alpha(b^\alpha) - cos_\alpha(b^\alpha)]\right] + 1 - t^2\left({}_0I_b^\alpha\right)[E_\alpha(-tx^\alpha) \otimes_\alpha sin_\alpha(x^\alpha)]. \tag{15}$$

Therefore,

$$(1+t^2)\left({}_0I_b^\alpha\right)[E_\alpha(-tx^\alpha) \otimes_\alpha sin_\alpha(x^\alpha)] = \left[E_\alpha(-tb^\alpha) \otimes [-t sin_\alpha(b^\alpha) - cos_\alpha(b^\alpha)]\right] + 1. \tag{16}$$

And hence,

$$\left({}_0I_b^\alpha\right)[E_\alpha(-tx^\alpha) \otimes_\alpha sin_\alpha(x^\alpha)] = \frac{E_\alpha(-tb^\alpha) \otimes_\alpha [-t sin_\alpha(b^\alpha) - cos_\alpha(b^\alpha)]}{1+t^2} + \frac{1}{1+t^2}.$$ q.e.d.

Theorem 3.2: *Let* $0 < \alpha \le 1$ *and* $t > 0$. *Then the improper* α*-fractional integral*

$$\left({}_0I^{\alpha}_{+\infty}\right)\left[E_\alpha(-tx^\alpha)\otimes_\alpha sin_\alpha(x^\alpha)\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(-1)}\right] = \frac{\pi}{2} - \arctan t\,. \tag{17}$$

Proof Let $F_\alpha(t) = \left({}_0I^{\alpha}_{+\infty}\right)\left[E_\alpha(-tx^\alpha)\otimes_\alpha sin_\alpha(x^\alpha)\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(-1)}\right]$, by differentiation under fractional integral sign, we obtain

$$\frac{d}{dt}F_\alpha(t)$$

$$= \frac{d}{dt}\left({}_0I^{\alpha}_{+\infty}\right)\left[E_\alpha(-tx^\alpha)\otimes_\alpha sin_\alpha(x^\alpha)\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(-1)}\right]$$

$$= \left({}_0I^{\alpha}_{+\infty}\right)\left[\frac{d}{dt}E_\alpha(-tx^\alpha)\otimes_\alpha sin_\alpha(x^\alpha)\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(-1)}\right]$$

$$= -\left({}_0I^{\alpha}_{+\infty}\right)\left[E_\alpha(-tx^\alpha)\otimes_\alpha sin_\alpha(x^\alpha)\right]$$

$$= -\lim_{b\to+\infty}\left({}_0I^{\alpha}_{b}\right)\left[E_\alpha(-tx^\alpha)\otimes_\alpha sin_\alpha(x^\alpha)\right]$$

$$= -\lim_{b\to+\infty}\left[\frac{E_\alpha(-tb^\alpha)\otimes_\alpha[-tsin_\alpha(b^\alpha)-cos_\alpha(b^\alpha)]}{1+t^2} + \frac{1}{1+t^2}\right] \quad \text{(by Lemma 3.1)}$$

$$= -\frac{1}{1+t^2}$$

Thus,

$$F_\alpha(q) - F_\alpha(p) = -\int_p^q \frac{1}{1+t^2}dt = \arctan p - \arctan q\,. \tag{18}$$

Since $\lim_{p\to+\infty} F_\alpha(p) = 0$ and $\lim_{p\to+\infty}\arctan p = \frac{\pi}{2}$, it follows that

$$F_\alpha(q) = \frac{\pi}{2} - \arctan q. \tag{19}$$

Thus,

$$\left({}_0I^{\alpha}_{+\infty}\right)\left[E_\alpha(-tx^\alpha)\otimes_\alpha sin_\alpha(x^\alpha)\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(-1)}\right] = \frac{\pi}{2} - \arctan t. \qquad \text{q.e.d.}$$

In Theorem 3.2, let $t = 0$, we have

Corollary 3.3: *If* $0 < \alpha \le 1$, *then*

$$\left({}_0I^{\alpha}_{+\infty}\right)\left[sin_\alpha(x^\alpha)\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(-1)}\right] = \frac{\pi}{2}\,. \tag{20}$$

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional calculus and a new multiplication of fractional analytic functions, some type of improper fractional integral is studied. Using differentiation under fractional integral sign and integration by parts for fractional calculus, we can find the exact solution of this improper fractional integral. In fact, our result is a generalization of classical calculus result. In the future, we will continue to use our methods to study the problems in fractional differential equations and engineering mathematics.

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